Stochastic volatility and option pricing

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Introduction
Assume that the stock price movement is,

\[ S_t = S_0 \exp \left( (\mu - \sigma^2/2)t + \sigma W_t \right), \]

where

- \( W_t \) is standard Brownian motion,
  \( W_t \sim N(0, t) \),

- \( \mu \) is the drift, i.e., \( E[S_t] = S_0 e^{\mu t} \) and

- \( \sigma \) is the volatility.

This model does not fit the empirical observations of log returns.
Features of log returns
Empirically found facts about the log returns,

• Sup-log-linear tails

• Aggregational Gaussianity for increasing time interval

• Uncorrelated over time

• Volatility clustering

• Skew in returns

• Leverage effect (negative return gives higher future level of volatility).
European Call Options

The payoff from a European call option is,

\[ C(S_T) = \max(S_T - K, 0), \]

where \( K \) is the strike price and \( T \) the time to maturity.

The option price is derived from arguments such as replicating portfolio and no arbitrage.

Any derivative can be priced as

\[ \Pi(S_0) = e^{-rT} E^Q [C], \]

where \( Q \) denotes a new measure (martingale measure) and \( r \) is the interest rate.
European Call Options

In the case of Geometric Brownian motion we have that

\[ S_t = S_0 \exp \left( (r - \sigma^2/2)t + \sigma W_t \right), \]

under \( Q \).

This gives the Black-Scholes formula

\[ \Pi(S_0, K, r, T, \sigma) = S_0 N(d_1) + e^{-rT} K N(d_2), \]

where

\[ d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma \sqrt{T}} \]

and

\[ d_2 = d_1 - \sigma \sqrt{T}. \]
Implied Volatility
We observe the price \( C_{\text{obs}} \) of a European call. The implied volatility, \( \sigma_I \), solves

\[
\Pi(S_0, K, \sigma_I) = C_{\text{obs}}(K).
\]

This is a convenient way to compare prices between different derivatives.

In theory there should be no difference in implied volatility between different derivatives with the same underlying commodity.

Empirically this is not true (smile and smirk effects).
Hull White

Assume that under $Q$

$$\sigma = \begin{cases} 
\sigma_1 & \text{with} & p \\
\sigma_2 & \text{with} & q = 1 - p. 
\end{cases}$$

This is a simple **Stochastic Volatility Model**.

The implied volatility in this model solves

$$\Pi(\sigma, K) = p\Pi(\sigma_1, K) + q\Pi(\sigma_2, K).$$
Hull White
Assume we have observed option prices $C_{\text{obs}}(K_i)$ at strike levels $K_i$. The implied parameters of the Hull-White model would then be set by solving

$$\min_{p, \sigma_1, \sigma_2} \sum_i (p \Pi(\sigma_1, K_i) + q \Pi(\sigma_2, K_i) - C_{\text{obs}}(K_i))^2.$$
**Hull White**

There are drawbacks of this model,

- It is too simple. (Larger sample space)
- Today-tomorrow model.
- Does not capture volatility clustering.
Option pricing

A common model to use is

\[ dS = \mu S dt + \sigma(t) S dW \]
\[ \sigma(t) = f(Y(t)) \]
\[ dY = (a + bY) dt + cY^\gamma dV \]
\[ dV = \rho dW + \sqrt{1 - \rho^2} dZ , \]

where \( W(t) \) and \( Z(t) \) are independent Brownian motions and \( \rho \) is a correlation coefficient. Usually \( \rho \approx -0.5 \).
**Mean reverting process**
Continuous time processes must be such that \( \sigma(t) \) is always non-negative.

**Example:**

\[
d \log(\sigma(t)) = a + b \log(\sigma(t))dt + \omega dW(t).
\]

**Example:** CIR type of model,

\[
d\sigma(t) = (a + b\sigma(t))dt + \omega\sqrt{\sigma(t)}dW(t).
\]

**Example:**

\[
d\sigma(t) = a + b\sigma(t)dt + \omega dW(t)
\]

not a good choice since \( \sigma(t) \) may be negative.

How should an option be priced?
Option pricing

The measure $Q$ is not unique, which means that more than one option price guarantees no arbitrage.

If there is another priced derivative then a unique price can be derived.
Option pricing

Fix $Q$, then option prices can be derived by:

- Simulation.

- If the mean reversion is fast, then one can expect that the price of an option is close to the mean of $\sigma(t)$.

This leads to an expansion of the option price in terms of the rate of mean reversion $b$ and a corrected price compared to Black-Scholes formula.

$$\Pi = \Pi_{BS} - T \left( V_2 s^2 \frac{d^2 \Pi}{ds^2} + V_3 s^3 \frac{d^3 \Pi}{ds^3} \right).$$

where $V_2$ and $V_3$ are functions of $b$. 
References

1. Fouque J.-P., et al. Derivatives in Financial Markets with Stochastic Volatility,

2. Barndorff-Nielsen O., Shephard N., Financial volatility and Lévy based models